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The relativistically rigid motion of a surface

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Abstract. The kinematics of the relativistically rigid motion of a surface is investigated in a Lorentzian manifold and the influence of the rigidity conditions on the congruence of the world lines of the points of that surface is examined.

1. Introduction

The superficial rigid surface considered by Pounder (1954) in special relativity, according to a definition of Synge (1954), has been called forth by Vigier (1979) to account for Aspect's experiments (1975) by introducing the superluminal propagation of the quantum potential so as to justify non-local interactions between measurements separated by space-like intervals.

However, the treatment of Synge is not manifestly covariant and results obtained are restricted to slow motions, whereas the motions discussed by Pounder 'are very special and relate rather to the question of corresponding rigid surfaces'. It is therefore our purpose to give here a general kinematical study of rigid motions of a surface in a Lorentzian manifold and to characterise the allowed types of motion.

We give in § 2 a local definition of the rigid motion of a surface Σ, in strict analogy with the Born (1909) criterion of rigidity of relativistic bodies, which recovers that of Synge. We examine in § 3 the properties of the time-like congruence $\mathscr{C}_u$ formed by the world lines of points of Σ. Some particular cases are also investigated. Concluding remarks follow in § 4.

2. Definition of superficial rigidity

Let us consider in a Lorentzian manifold of metric g_{ab} the time-like congruence $\mathscr{C}_u$ formed by the world lines tangent to the four-velocities u^a of points of a surface Σ:

$$u^a u_a = -1. \tag{1}$$

The connecting vector η^a of $\mathscr{C}_u$ satisfies

$$\mathscr{L}_u \eta^a = u^b \eta^a_{;b} - \eta^b u^a_{;b} \tag{2}$$

where $\mathscr{L}_u$ is the Lie derivative with respect to u^a and the semicolon stands for covariant differentiation. On the other hand, the hypersurface $F(x^a)$ generated by Σ and $\mathscr{C}_u$ is such that

$$\mathscr{L}_u F(x^a) = u^b\, \partial_b F(x^a) = 0, \qquad \partial_b = \partial/\partial x^b. \tag{3}$$

The vector field n^a defined by

$$n_a = \partial_a F(x)/|\partial_a F(x)| \tag{4a}$$

satisfies

$$n^a n_a = 1, \qquad u^a n_a = 0. \tag{4b}$$

Let h_a^b (resp f_a^b) be the projection operator onto the hyperplane Π_u normal to u^a (resp the plane $P_{u,n}$ normal to u^a and n^a),

$$h_a^b = \delta_a^b + u_a u^b, \qquad f_a^b = h_a^b - n_a n^b. \tag{5a, b}$$

The tensors $h_{ab} = h_a^c h_b^d g_{cd}$ and $f_{ab} = f_a^c f_b^d g_{cd}$ play the role of the metric in Π_u and $P_{u,n}$ respectively; they serve to define the orthogonal distance of two world lines, i.e. the length of the projection of the connecting vector either in Π_u or in $P_{u,n}$.

In accordance with Born and Synge, the motion of a surface Σ is said to be rigid if the orthogonal distance of the two world lines of two arbitrary neighbouring points of Σ is invariant during the motion:

$$\mathscr{L}_u (f_{ab} \eta^a \eta^b) = 0 \tag{6}$$

which gives on account of (2)

$$\mathscr{L}_u f_{ab} = u_{a;b} + u_{b;a} + \dot{u}_a u_b + \dot{u}_b u_a - n_a \mathscr{L}_u n_b - n_b \mathscr{L}_u n_a = 0 \tag{7}$$

where $\dot{u}_a = u^b u_{a;b}$. These ten equations are not however all independent; we have identically, on account of (4),

$$u^b \mathscr{L}_u f_{ab} \equiv 0, \qquad n^a n^b \mathscr{L}_u f_{ab} \equiv 0. \tag{8a, b}$$

Thus, only five of the equations (7) are independent and may serve to determine the five independent components of u^a and n^a constrained by $u^a u_a = -1 = -n^a n_a$ and $u^a n_a = 0$; in consequence, the usual degrees of freedom of the surface all remain available.

On the other hand, we have

$$n^b \mathscr{L}_u f_{ab} \equiv -g_{ab} \mathscr{L}_u n^b + n^b \dot{u}_b u_a - n^b \mathscr{L}_u n_b n_a = 0, \tag{9a}$$

whence

$$\mathscr{L}_u n^a = A u^a + B n^a, \qquad A = n^b \dot{u}_b, \qquad B = -n^a n^b u_{a;b}. \tag{9b}$$

Therefore during the rigid motion of Σ the vectors u^a and n^a are two-surface forming.

On the other hand, we have

$$\mathscr{L}_u n_a = -B n_a \tag{10}$$

as, on account of the hypersurface orthogonality of n^a,

$$n_{[a;b} n_{c]} = 0 \tag{11a}$$

which gives, after multiplication by $u^b n^c$,

$$2 n_{[a;b]} u^b = u^b n_{a;b} + n_b u^b{}_{;a} = n^b n^c u_{b;c} n_a. \tag{11b}$$

3. Properties of $\mathscr{C}_u$

Now, we introduce the decomposition of $u_{a;b}$ into the shear σ_{ab}, the expansion θ, the vorticity ω_{ab} of $\mathscr{C}_u$ by

$$u_{a;b} = \sigma_{ab} + \theta h_{ab} + \omega_{ab} - \dot{u}_a u_b \tag{12a}$$

with

$$\sigma_{ab} u^b = 0 = \omega_{ab} u^b, \qquad \sigma^a_a = 0, \qquad \theta = \tfrac{1}{3} u^a_{;a}. \tag{12b}$$

By contraction equation (7) gives

$$B + 3\theta = 0 \tag{13}$$

and

$$\sigma_{ab} = \tfrac{1}{2}(n_a \mathscr{L}_u n_b + n_b \mathscr{L}_u n_a) - \theta h_{ab} \tag{14}$$

whence, on account of equations (10) and (13),

$$\sigma_{ab} = 2\theta n_a n_b - \theta f_{ab}. \tag{15}$$

Thus n^a is a shear eigenvector and the overall expansion θ of $\mathscr{C}_u$ is counterbalanced by $\tilde{\sigma}_{ab} = f^c_a f^d_b \sigma_{cd}$ on the surface, so that the motion of Σ is rigid.

As for the vorticity vector ω^a,

$$\omega^a = \tfrac{1}{2} \eta^{abcd} u_b \omega_{cd}, \tag{16}$$

it remains arbitrary. In general the unit vector $r^a = \omega^a / |\omega^a|$ is not tangent to $F(x^a)$; however, when $r^a n_a = 0$, we may define a vorticity congruence and extend to it Greenberg's results (1970).

Special cases of interest are obtained when r^a is Fermi transported ($\dot{r}^a = r^b \dot{u}_b u^a$) or Lie dragged ($\mathscr{L}_u r^a = 0$).

Remark 1. The condition for the vectors orthogonal to u^a in the hypersurface $F(x^a)$ generated by Σ and $\mathscr{C}_u$ to be two-surface forming is

$$p^b q^a_{;b} - q^b p^a_{;b} = \alpha p^a + \beta q^a \tag{17}$$

for any two independent vectors p^a and q^a tangent to $F(x^a)$ and orthogonal to u^a. This implies

$$f^c_a f^d_b u_{[c;d]} = 0 \Leftrightarrow \omega^a n_a = 0 \tag{18a}$$

and

$$f^c_a f^d_b n_{[c;d]} = 0 \tag{18b}$$

which is identically satisfied as n^a is hypersurface orthogonal.

Remark 2. A superficially rigid congruence $\mathscr{C}_u$ may be said, following Ehlers (1961), to be superficially isometric when u^a is parallel to a Killing vector of the hypersurface $F(x^a)$, i.e.

$$\mathscr{L}_U (g_{ab} - n_a n_b) = 0, \qquad U^a = e^{\lambda} u^a. \tag{19}$$

The comparison of equations (7) and (19) gives

$$\dot{u}_a = \partial_a \lambda, \qquad \dot{\lambda} = 0. \tag{20}$$

Thus, the congruence $\mathscr{C}_u$ is superficially isometric if and only if

$$\dot{u}_{[a;b]} = 0. \tag{21}$$

4. Concluding remarks

It appears therefore that superficial rigidity restricts the possible types of motion of a surface Σ by relating the shear and the expansion of the rigid congruence by equation (15). It does not imply, as in the Newtonian case, the nullity of those quantities.

The particular motions considered by Pounder may be obtained by the following additional requirements: Σ is a surface of revolution and the vorticity unit vector r^a lying along the axis of revolution is Lie dragged.

As for the superluminal propagation of the quantum potential contemplated by Vigier, it does not follow directly from the above considerations; it requires further investigations.

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